

Spatial entanglement of twin quantum images

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Abstract

We show that spatial entanglement of two twin images obtained by parametric down-conversion is complete, i.e. concerns both amplitude and phase. This is realised through a homodyne detection of these images which allows for measurement of the field quadrature components. EPR correlations are shown to exist between symmetrical pixels of the two images. The best possible correlation is obtained by adjusting the phase of the local oscillator field (LO) in the area of maximal amplification. The results for quadrature components hold unchanged even in absence of any input image i.e. for pure parametric fluorescence. In this case they are not related to intensity and phase fluctuations.

I. INTRODUCTION

Optical systems which display quantum entanglement properties in the spatial domain are of great interest for applications, since the amount of information that can be manipulated and processed in parallel exploiting quantum correlation effects increases substantially with respect to the case of single mode beams. Recently, there has been a rise of interest in the utilisation of entangled beam in optical imaging (quantum imaging) [1, 2, 3, 4].

In this paper, we consider the field generated through the process of frequency down-conversion in a travelling wave optical parametric amplifier (OPA). In [2, 4] it was demonstrated that such a system, when coupled with an appropriate classical imaging device, is able to generate two symmetrical amplified copies of an injected input image that are strongly correlated one to each other: they indeed display synchronized local intensity fluctuations at the level of quantum noise, and for this reason they may be referred to as twin images.

Here we present new results, that consolidate and complete the picture, showing that the two output images are locally correlated, not only with respect to intensity fluctuations, but also to “phase” fluctuations. To carry out a phase-sensitive measurements, we consider a homodyne detection scheme that allows us to compare the fluctuations of field quadrature components from two corresponding pixels of the two output images. We find in general that, for an arbitrary quadrature component characterized by the phase ϕ_L of the local oscillator, the difference between the fluctuations measured in two symmetrical pixels displays exactly the same spectrum as the sum of the fluctuations in the orthogonal quadrature component $\phi_L + \pi/2$. The common value can be reduced well below the shot noise level over the whole image area, provided that the amplification is large enough and the phase ϕ_L is correctly adjusted. The choice of the phase is crucial to obtain a large level of quantum correlations between symmetrical pixels for the quadrature component ϕ_L and anticorrelation for the quadrature component $\phi_L + \pi/2$. Thus, the twin images exhibit a complete Einstein-Podolsky-Rosen (EPR) [5] entanglement with respect to continuous variables [6].

Since quantum correlations are shown to exist for any couple of symmetrical pixels over the whole area of the output images, we speak of spatial quantum entanglement. The system exhibits a *spatial* realization of the EPR paradox for two orthogonal quadrature components of the output field similar to that shown in [7] for the case of the parametric oscillator below

threshold. In comparison with the analysis of [7], we consider here also the case in which a coherent image is injected into the system. Furthermore, the consideration of an OPA, instead of an optical parametric oscillator with spherical mirrors, allows for obtaining here completely analytical results.

In presence of an input image, the mean output field is different from zero, and therefore amplitude (i.e. intensity) and phase fluctuations correspond to special cases of quadrature fluctuations. Therefore the previous analysis allows for concluding in a rather straightforward way that in symmetrical pixels not only quantum intensity fluctuations are strongly correlated, but phase fluctuations are anticorrelated in the same amount.

The paper is divided as follows. After a presentation of the optical image amplification scheme in section II, in the third section we study the fluctuation spectrum of the quadrature components measured in homodyne detection. The fourth section is devoted to the discussion of amplitude and phase fluctuations. The final section includes conclusions and perspectives.

II. OPTICAL IMAGE AMPLIFICATION SCHEME

The experimental procedure to generate a pair of quantum entangled images through the process of parametric down-conversion close to the degenerate frequency has been discussed in previous papers [2, 4] and can be summarized as follows. The $\chi^{(2)}$ crystal is enclosed between two lenses L and L' , as shown in Fig.I. We take the z axis as the main light propagation direction and indicate with $\vec{x} = (x, y)$ the point coordinates in a generic transverse plane. Not shown in the figure is the coherent pump field that activates the process of down-conversion and which we take as an ideal classical monochromatic plane wave of frequency ω_p , propagating inside the crystal along the z direction.

The crystal slab of width l_c , ideally infinite in the transverse directions, is cut for type I quasi-collinear phase-matching at the degenerate frequency $\omega_p/2$. Under these assumptions, each elementary down-conversion process corresponds to the splitting of a pump photon of frequency ω_p into a pair of photons of frequencies $\omega_p/2 + \Omega$ and $\omega_p/2 - \Omega$ (with $\Omega \ll \omega_p$), propagating with the same polarization and with opposite transverse wavevectors \vec{q} and $-\vec{q}$.

We designate by $a_1(\vec{x}, t)$, $a_2(\vec{x}, t)$, $a_3(\vec{x}, t)$, $a_4(\vec{x}, t)$, the slowly varying envelope operators of the down-converted field (with respect to the carrier frequency $\omega_p/2$) in the input plane

P_1 , the entrance plane P_2 of the $\chi^{(2)}$ crystal slab, its exit plane P_3 , and the output plane P_4 , respectively (Fig. 1). We denote by $a_i(\vec{x}, \Omega)$, $a_i(\vec{q}, \Omega)$ ($i = 1, \dots, 4$) their Fourier transforms in time and in space-time respectively. The purpose of the two lenses is to map the Fourier plane (q_x, q_y) into the physical plane (x, y) . In this manner, if an optical image is injected at the degenerate frequency $\omega_p/2$ in the object plane P_1 , the system amplifies portions of this image rather than a band of its q -vectors. Indeed, the input-output transformation which describes propagation inside the crystal in the linear regime, assuming that pump depletion and losses are negligible, can be written as [8]:

$$a_3(\vec{q}, \Omega) = u(\vec{q}, \Omega)a_2(\vec{q}, \Omega) + v(\vec{q}, \Omega)a_2^\dagger(-\vec{q}, -\Omega), \quad (1)$$

The presence of the lenses converts it into a relation between the real-space field operators in the plane P_1 and P_4 :

$$a_4(\vec{x}, \Omega) = \bar{u}(\vec{x}, \Omega)a_1(-\vec{x}, \Omega) + \bar{v}(\vec{x}, \Omega)a_1^\dagger(\vec{x}, -\Omega), \quad (2)$$

where

$$\bar{u}(\vec{x}, \Omega) = -u\left(\frac{2\pi\vec{x}}{\lambda f}, \Omega\right), \quad \bar{v}(\vec{x}, \Omega) = v\left(\frac{2\pi\vec{x}}{\lambda f}, \Omega\right), \quad (3)$$

f is the focal length of the two lenses and λ is the wavelength of the down-converted field. The explicit expressions of the gain coefficients $u(\vec{q}, \Omega)$ and $v(\vec{q}, \Omega)$ can be found in [9]. Here we just notice that they depend on the linear gain parameter σ and on the dispersion properties of the crystal; they are functions of the modulus of \vec{q} and Ω and, for $\Omega = 0$, display a broad maximum, corresponding to that transverse wavenumber which is phase-matched at the degeneracy frequency, for which (within the paraxial approximation)

$$q^2 = k_s^2 - (k_p/2)^2 \approx k_s \Delta_0, \quad (4)$$

where k_s , k_p are the wave numbers of the signal and the pump field at the carrier frequencies $\omega_p/2$ and ω_p , respectively, and $\Delta_0 = 2k_s - k_p$ is the collinear phase mismatch parameter which is assumed non negative. The width of the *plateau* around the value (4) is on the order of $q_0 = \sqrt{k_s/l_c}$, the variation scale of $|u|$ and $|v|$ in the spatial frequency domain.

We underline that all the results that follow do not depend on the particular form of the gain functions, but rely on the fact that they satisfy the following unitarity conditions

$$|u(\vec{q}, \Omega)|^2 - |v(\vec{q}, \Omega)|^2 = 1, \quad (5)$$

$$u(\vec{q}, \Omega)v(-\vec{q}, -\Omega) = u(-\vec{q}, -\Omega)v(\vec{q}, \Omega),$$

which guarantee that the free field commutation rules are preserved:

$$\begin{aligned} [a_i(\vec{q}, \Omega), a_i^\dagger(\vec{q}', \Omega')] &= \delta(\vec{q} - \vec{q}')\delta(\Omega - \Omega'), \\ [a_i(\vec{q}, t), a_i(\vec{q}', t')] &= 0. \quad (i = 1, 2, 3, 4) \end{aligned} \quad (6)$$

On the other hand, with respect to other systems which exhibit input/output relations of the same form (e.g. optical parametric oscillators, see e.g. [10]), the large spatial bandwidth q_0 of the amplifier makes this travelling-wave scheme a good candidate for high resolution image amplification.

For the scheme of Fig.1, the region in the transverse plane which can be efficiently amplified without distortion has a linear size on the order of

$$x_0 = \frac{\lambda f}{2\pi} q_0, \quad (7)$$

which represents the width of the *plateau* of the real-space gain functions (3). Such a region has either the shape of a disc of area $\sim S_0 = x_0^2$ centered at the origin, or a ring of width $\sim x_0$, depending on the possibility to have collinear ($\Delta_0 = 0$) or non-collinear ($\Delta_0 > 0$) phase-matching at $\Omega = 0$, respectively. We assume that the input image is a coherent stationary field of frequency $\omega_p/2$ confined in this region of plane P_1 (see Fig.1) so that

$$\langle a_1(\vec{x}, \Omega) \rangle = \sqrt{2\pi}\delta(\Omega)\alpha_{in}(\vec{x}). \quad (8)$$

As explained in details in [4, 8], whenever the input image is symmetric with respect to the system axis, the device works as a phase-sensitive amplifier (see in this connection also [11]). In this case, the phase of the input image must be selected in order to optimize the gain. Assuming the input image is duplicated before amplification by means of a classical imaging device which allows to obtain a symmetrical field distribution (i.e. $\alpha_{in}(-\vec{x}) = \alpha_{in}(\vec{x})$), the system is able to generate in the output plane two amplified copies that are far better correlated in space-time than the originals, meaning by this that they display perfectly (in the ideal case) synchronized local intensity fluctuations. It was also demonstrated [4] that in the limit of high gain, the signal-to-noise ratio as measured from a small portion of the input image before duplication is preserved in the corresponding portions of the two output images: *noiseless* amplification is therefore achieved for both output channels taken separately (see [12] for an experimental observation of noiseless amplification of images).

In [2] an alternative way to generate a pair of quantum correlated images (also called *twin images*) was considered; it consists in injecting a single input image asymmetrically, for

example by confining it to the upper half of plane P_1 as shown in Fig.1. This configuration does not require a duplication system and presents the further advantage that the gain does not depend on the phase of the input field because the system works as a phase-insensitive amplifier. However, the fidelity with which information is transferred is worse than in the phase-sensitive case, since the signal-to-noise ratio is deteriorated at least by a factor two in the amplification process (a feature common to all phase-insensitive optical amplifiers [13]).

Most of the results presented in this paper do not depend on the particular injection scheme, so no assumption are made on the input intensity distribution $|\alpha_{in}(\vec{x})|^2$. Imperfect detection can be modelled in the usual way, by coupling the output field operator $a_4(\vec{x}, t)$ with an independent operator field $a_N(\vec{x}, t)$ which acts on the vacuum state. The contribution a_N describes the noise added by losses in the detection process; thus the effective output field measured by a detector of quantum efficiency $\eta \leq 1$ is

$$a_D(\vec{x}, t) = \sqrt{\eta}a_4(\vec{x}, t) + \sqrt{1 - \eta}a_N(\vec{x}, t) \quad (9)$$

and the corresponding photon flux density is

$$i(\vec{x}, t) = a_D^\dagger(\vec{x}, t)a_D(\vec{x}, t) . \quad (10)$$

As shown in Fig.1, at the exit face of the crystal we insert a pupil of area S_p , an element that allows for eliminating divergencies which arise in the calculation of the field mean intensity and correlation functions when dealing with a system of infinite transverse dimensions [8, 10]. It also determines the characteristic resolution area of the device in the detection plane, which is $S_R = (\lambda f)^2/S_p$. This finite size optical element introduces a convolution integral with the pupil response function in the r.h.s. of Eq. (2) and, as a consequence, the points of the input image are spread into diffraction spots of area S_R in the output image. However, analytical calculations are performed in the *low diffraction limit*, assuming that the diffraction spot size $\sqrt{S_R}$ is much smaller than both x_0 and the variation scale of the input image intensity. Considering a single pixel detector (labelled by index j) that intercepts the photons arriving on an area R_j which is large in comparison with S_R , the mean value of the measured photocurrent is then [4]:

$$\begin{aligned} \langle i_j(t) \rangle &= \int_{R_j} d\vec{x} \langle i(\vec{x}, t) \rangle \\ &= \eta \int_{R_j} d\vec{x} |\overline{u}(\vec{x}, 0)\alpha_{in}(-\vec{x}) + \overline{v}(\vec{x}, 0)\alpha_{in}^*(\vec{x})|^2 + \frac{\eta}{S_R} \int_{R_j} d\vec{x} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} |\overline{v}(\vec{x}, \Omega')|^2 . \end{aligned} \quad (11)$$

The first integral represents the amplified coherent input field while the second integral is the contribution coming from spontaneous parametric down-conversion. The ratio S_0/S_R gives an evaluation of the number of details of the input image which can be resolved in the detection plane (e.g. with the pixel array of a CCD camera). Moreover, quantum correlation effects tends to disappear when $S_R \rightarrow S_0$, since in this limit the signal and idler photons of each down-converted pair can no more be resolved separately, because of the large diffraction spread in q -space. Making S_R as small as possible with respect to S_0 is therefore a necessary requirement that must be taken into account in experiments. However, this leads to an increase of the spontaneous emission contribution which goes at the expense of the visibility of the amplified input image. This last circumstance imposes a lower limit on the intensity of the input image (see [2, 8, 10] for more details).

III. CORRELATIONS MEASUREMENT IN A HOMODYNE DETECTION SCHEME

A homodyne detection scheme allows for the measurement of a particular quadrature component of the field. It consists in a beam splitter that combines the output field with a coherent field of much higher intensity, $\alpha_L(\vec{x})$, which can be treated as a classical quantity and is usually referred to as the local oscillator field (LO). In the balanced version a 50/50 beam splitter is used, so that the operators associated to the fields coming from the two output ports of the beam-splitter, labelled by b and c , are

$$a^{b,c}(\vec{x}, t) = [a_4(\vec{x}, t) \pm \alpha_L(\vec{x})] / \sqrt{2} \quad (12)$$

and the effective fields seen by two identical detectors of quantum efficiency η in the two ports b and c are

$$a_D^{b,c}(\vec{x}, t) = \sqrt{\eta} a^{b,c}(\vec{x}, t) + \sqrt{1 - \eta} a_N^{b,c}(\vec{x}, t) , \quad (13)$$

where $a_N^{b,c}(\vec{x}, t)$ describe the noise added in the detection process. When the corresponding intensities are electronically substracted, one obtains a direct measure of the quadrature component of the output field a_4 selected by the phase of the LO, more precisely

$$Z_{\phi_L}(\vec{x}, t) = a_D^{b\dagger}(\vec{x}, t) a_D^b(\vec{x}, t) - a_D^{c\dagger}(\vec{x}, t) a_D^c(\vec{x}, t) \quad (14)$$

$$\xrightarrow{\eta \rightarrow 1} \rho_L(\vec{x}) \left[a_4^\dagger(\vec{x}, t) e^{i\phi_L(\vec{x})} + a_4(\vec{x}, t) e^{-i\phi_L(\vec{x})} \right] , \quad (15)$$

where $\rho_L(\vec{x}) = |\alpha_L(\vec{x})|$ and $\phi_L(\vec{x}) = \arg \alpha_L(\vec{x})$. Taking into account the finite size of the pixel detection area R_j , the measured quantity is

$$Z_{\phi_L}^{(j)}(t) = \int_{R_j} d\vec{x} Z_{\phi_L}(\vec{x}, t) . \quad (16)$$

We now want to compare the fluctuations of the field quadrature measured in two symmetrical pixels $j = 1$ and $j = 2$ of the signal and idler image. To this aim, we consider the sum and the difference of the quadrature obtained from two symmetrical detection regions R_1 and R_2 :

$$Z_{\phi_L}^{(\pm)}(t) = Z_{\phi_L}^{(1)}(t) \pm Z_{\phi_L}^{(2)}(t) . \quad (17)$$

The corresponding fluctuation spectra, defined as

$$\begin{aligned} V_{\phi_L}^{(\pm)}(\Omega) &= \int_{-\infty}^{\infty} dt e^{i\Omega t} \langle \delta Z_{\phi_L}^{(\pm)}(t) \delta Z_{\phi_L}^{(\pm)}(0) \rangle , \\ \delta Z_{\phi_L}^{(\pm)}(t) &= Z_{\phi_L}^{(\pm)}(t) - \langle Z_{\phi_L}^{(\pm)}(t) \rangle , \end{aligned} \quad (18)$$

describe the degree of correlation between the observables $Z_{\phi_L}^{(1)}$ and $Z_{\phi_L}^{(2)}$. Using the input-output transformation (2), the commutation rules (6) and the fact that the input image is coherent we obtain the following relations:

$$\begin{aligned} V_{\phi_L}^{(-)}(\Omega) &= V_{\phi_L + \pi/2}^{(+)}(\Omega) \\ &= (1 - \eta)(SN)_{LO} + \eta^2 \int_{R_1 + R_2} d\vec{x} |\bar{u}(\vec{x}, \Omega) \alpha_L^*(\vec{x}) - \bar{v}^*(\vec{x}, \Omega) \alpha_L(-\vec{x})|^2 , \end{aligned} \quad (19)$$

where

$$(SN)_{LO} = \eta \int_{R_1 + R_2} d\vec{x} \rho_L^2(\vec{x}) \quad (20)$$

is the shot noise level determined by the LO on the two detectors (we assumed $|\alpha_L(\vec{x})|^2 \gg \langle j(\vec{x}, t) \rangle$). Next, we assume that the LO is symmetric with respect to the system axis, i.e. $\alpha_L(\vec{x}) = \alpha_L(-\vec{x})$. Because $\bar{v}(\vec{x}, \Omega) = \bar{v}(\vec{x}, -\Omega)$ we can write:

$$V_{\phi_L}^{(-)}(\Omega) = V_{\phi_L + \pi/2}^{(+)}(\Omega) = (1 - \eta)(SN)_{LO} + \eta^2 \int_{R_1 + R_2} d\vec{x} F(\vec{x}, \Omega) \rho_L^2(\vec{x}) \quad (21)$$

where

$$F(\vec{x}, \Omega) = |\bar{u}(\vec{x}, \Omega) e^{-i\phi_L(\vec{x})} - \bar{v}^*(\vec{x}, \Omega) e^{i\phi_L(\vec{x})}|^2 . \quad (22)$$

Note that the above expression corresponds to the fluctuation spectrum normalized to shot noise when $\eta = 1$ and the pixel area is small with respect to x_0^2 and to the square of the scale

of variation of $\alpha(\vec{x})$. In this case \vec{x} in Eq.(21) must be taken as the central point of pixel 1 or pixel 2; the result is the same for both pixels because $\overline{u}(\vec{x}, \Omega) = \overline{u}(-\vec{x}, \Omega)$, $\overline{v}(\vec{x}, \Omega) = \overline{v}(\vec{x}, \Omega)$ and $\phi_L(\vec{x}) = \phi_L(-\vec{x})$.

A first important result follows from the first equality (19)), according to which $Z_{\phi_L}^{(1)}$ and $Z_{\phi_L}^{(2)}$ are correlated one to each other exactly to the same extent as the corresponding orthogonal quadrature components $Z_{\phi_L+\pi/2}^{(1)}$ and $Z_{\phi_L+\pi/2}^{(2)}$ are anti-correlated. Second, the common fluctuation spectrum of the two observables $Z_{\phi_L}^{(1)} - Z_{\phi_L}^{(2)}$ and $Z_{\phi_L+\pi/2}^{(1)} + Z_{\phi_L+\pi/2}^{(2)}$ as given by expression (19) does not depend on the intensity and phase of the input image. Hence the result is the same in the phase-insensitive and in the phase-sensitive scheme, and remains the same even in absence of an input image at all, i.e. in the case of pure parametric fluorescence. Third, this spectrum can be reduced well below the shot noise level, provided the gain is large enough and the phase of the LO is correctly adjusted. Indeed, assuming that

$$\phi_L(\vec{x}) = \frac{1}{2} (\arg \overline{u}(\vec{x}, 0) + \arg \overline{v}(\vec{x}, 0)) = \phi_{opt}(\vec{x}) \quad (23)$$

over the two detection areas, using the symmetry property of the LO $\alpha_L(-\vec{x}) = \alpha_L(\vec{x})$ and unitarity relations (5), one obtains from Eq. (19) for $\Omega = 0$:

$$F(\vec{x}, \Omega = 0) = \frac{1}{[|\overline{u}(\vec{x}, 0)| + |\overline{v}(\vec{x}, 0)|]^2}, \quad (24)$$

which goes to zero when $|\overline{u}(\vec{x}, 0)| \sim |\overline{v}(\vec{x}, 0)| \gg 1$. Under conditions of large gain and reasonably large quantum efficiency, almost perfect correlation between the selected quadratures can therefore be obtained.

It is interesting to relate the phase of optimum squeezing (in $Z_{\phi_L}^{(1)} - Z_{\phi_L}^{(2)}$) ϕ_{opt} for the LO with the phase of maximum amplification in the phase sensitive configuration. The mean output field is in general

$$\alpha_{out}(\vec{x}) = \langle a_4(\vec{x}, t) \rangle = \overline{u}(\vec{x}, 0) \alpha_{in}(-\vec{x}) + \overline{v}(\vec{x}, 0) \alpha_{in}^*(\vec{x}), \quad (25)$$

where we used Eq.(2) and Eq.(8). In the phase sensitive case $\alpha_{in}(-\vec{x}) = \alpha_{in}(\vec{x})$ we can write

$$|\alpha_{out}(\vec{x})|^2 = G(\vec{x}) |\alpha_{in}(\vec{x})|^2 \quad (26)$$

with the phase-sensitive gain given by:

$$G(\vec{x}) = |\overline{u}(\vec{x}, 0) e^{\phi_{in}(\vec{x})} + \overline{v}(\vec{x}, 0) e^{-\phi_{in}(\vec{x})}|^2, \quad (27)$$

where $\phi_{in}(\vec{x})$ is the phase of $\alpha_{in}(\vec{x})$. We easily obtain that the maximum gain

$$G_{max}(\vec{x}) = ||\bar{u}(\vec{x}, 0)| + |\bar{v}(\vec{x}, 0)||^2 \quad (28)$$

is obtained for:

$$\phi_{in}(\vec{x}) = \phi_{in}^{max}(\vec{x}) = \frac{1}{2} (\arg \bar{u}(\vec{x}, 0) - \arg \bar{v}(\vec{x}, 0)) . \quad (29)$$

On the other hand, from Eq.(25) with $\alpha_{in}(-\vec{x}) = \alpha_{in}(\vec{x})$ one obtains that when $\phi_{in} = \phi_{in}^{max}$ the phase ϕ_{out} of the output field α_{out} is given by

$$\phi_{out}(\vec{x}) = \phi_{out}^{max}(\vec{x}) = \frac{1}{2} (\arg \bar{u}(\vec{x}, 0) + \arg \bar{v}(\vec{x}, 0)) \quad (30)$$

and therefore coincides with $\phi_{opt}(\vec{x})$ given by Eq.(23). This leads to the following interpretation for ϕ_{opt} : the phase ϕ_L of optimum squeezing in $Z_{\phi_L}^{(-)}$ coincides with the phase of the output field in the phase-sensitive configuration, provided the phase of the input field is selected to have maximal amplification. Note that in the special case of perfect phase matching one has $\arg \bar{u}(\vec{x}, 0) = 0$ so that $\phi_{in}^{max}(\vec{x})$ given by Eq.(29) coincides with the corresponding $\phi_{out}(\vec{x})$.

The results obtained for the observables $Z_{\phi_L}^{(\pm)}$ closely resemble the situation of the EPR paradox for continuous variables demonstrated in [6], but generalised to many pixels (see also [7]) and to the presence of input images. We notice indeed that the conjugated observables $X_j = \int_{-T_D/2}^{T_D/2} dt Z_{\phi_L}^{(j)}(t)$ and $P_j = \int_{-T_D/2}^{T_D/2} dt Z_{\phi_L+\pi/2}^{(j)}(t)$ ($j = 1, 2$) obey the uncertainty rule:

$$\langle \delta^2 X_j \rangle \langle \delta^2 P_j \rangle \geq \frac{1}{4} \left[T_D \int_{R_1+R_2} d\vec{x} \rho_L^2(\vec{x}) \right]^2 . \quad (31)$$

On the other hand, the following combination over the two pixels: $X_- = X_1 - X_2$ and $P_+ = P_1 + P_2$ are commuting observables that can be simultaneously determined. When the time of measurement T_D is much larger than the inverse of the temporal bandwidth of the OPA, using Eq.(18), the uncertainty of these observables can be directly related to the fluctuations spectrum $V_{\phi_L}^{(-)}$

$$\langle \delta^2 X_- \rangle = \langle \delta^2 P_+ \rangle = T_D V_{\phi_L}^{(-)}(\Omega = 0) . \quad (32)$$

For $\eta = 1$, an optimal ajustement of the LO phase allows these uncertainties to reach almost a zero value for large amplification and thus to display an apparent violation of the Heisenberg rule:

$$\langle \delta^2 X_- \rangle \langle \delta^2 P_+ \rangle < \frac{1}{4} \left[T_D \int_{R_1+R_2} d\vec{x} \rho_L^2(\vec{x}) \right]^2 . \quad (33)$$

However, it is impractical to synthesize a LO with the phase variation prescribed by Eq.(23). On the other hand, for a LO with constant phase, the condition (23) concerning the phase of the LO can be exactly satisfied only for a single couple of pixels of area small compared to S_0 , so that the gain functions $|\bar{u}(\vec{x}, 0)|$ and $|\bar{v}(\vec{x}, 0)|$ are nearly uniform over the detection areas. We can however show that, by introducing an appropriate curvature to the wavefront of the LO field, EPR-like correlations are present for each couple of symmetric pixels in the output over the whole gain region S_0 . To this end, we allow the LO phase distribution to have a quadratic dependence on the spatial coordinate (which corresponds to a spherical wavefront as one has e.g. in gaussian beams). The wavefront curvature is selected in order to have the best fit of the spatial dependence of $\phi_{opt}(\vec{x})$ in Eq.(23).

Figure 2 plots the function $F(\vec{x}, \Omega = 0)$ in the limit where R_1 and R_2 are small compared to S_0 and symmetric. The collinear phase-mismatch at degeneracy is $\Delta_0 l_c = 0.5$ and the linear gain parameter is $|\sigma|l_c = 1.5$.

Curve (a) corresponds to the ideal case, with ϕ_L satisfying condition (23) everywhere in the transverse plane, and leads to a maximal amount of noise reduction in the whole amplification region. In curve (b), the phase of the LO is constant and satisfies condition (23) only in the point of maximum gain x_G , where perfect phase-matching is achieved, $|\bar{u}|^2 \simeq 5.5$ and $F = (|\bar{u}| - |\bar{v}|)^2 = \exp(-2|\sigma|l_c) \simeq 0.1$. Curve (c), obtained by optimizing the phase with a quadratic term (i.e. we take the form $\phi_L(\vec{x}) = \Phi_0 + \Phi_2(|\vec{x}| - x_G)^2/x_0^2$), is the best that can be done with a gaussian LO and is close to the ideal case.

IV. PHASE-INTENSITY ENTANGLEMENT OF THE TWIN IMAGES

Although the phase-sensitive measurement scheme considered in the last section offers a picture of the spatial correlations that can be observed in the output field, intensity correlation measurements are more straightforward to perform experimentally and lead also to interesting effects of quantum noise reduction [2, 4, 14]. The observable that displays reduced fluctuations with respect to the coherent state level is the difference between the direct photocurrents measured from two symmetrical detection region $i_- = i_1 - i_2$. The corresponding fluctuation spectrum is

$$V_{i_-}(\Omega) = \int_{-\infty}^{\infty} dt e^{i\Omega t} \langle \delta i_-(t) \delta i_-(0) \rangle \quad (34)$$

By using (2), (6), (9), (10), (11) and the fact that input image is in a coherent state one obtains after lengthy but elementary calculations

$$\begin{aligned}
V_{i_-}(\Omega) = & (1 - \eta)\langle i_+ \rangle \\
& + \eta^2 \int_{R_1+R_2} d\vec{x} |\bar{u}(\vec{x}, \Omega)\alpha_{out}^*(\vec{x}) - \bar{v}^*(\vec{x}, -\Omega)\alpha_{out}(-\vec{x})|^2 \\
& + \eta^2 \frac{1}{S_R} \int_{R_1+R_2} d\vec{x} \int \frac{d\Omega'}{2\pi} [|\bar{u}(\vec{x}, \Omega')|^2 |\bar{v}(\vec{x}, \Omega + \Omega')|^2 \\
& - \bar{u}(\vec{x}, \Omega + \Omega')\bar{v}(\vec{x}, -\Omega - \Omega')\bar{u}^*(\vec{x}, \Omega')\bar{v}^*(\vec{x}, -\Omega')] \quad (35)
\end{aligned}$$

where $\alpha_{out}(\vec{x})$ is given by Eq.(25). The shot noise level corresponds to the photocurrent sum $\langle i_+ \rangle = \langle i_1 + i_2 \rangle$. The second term on the l.h.s. of Eq. (35) arises from the interference between the amplified input field and the fluorescence field. The last term, which does not depend on the presence of an input field, is a pure noise contribution due to the self-interference of the fluorescence field and reduces to zero for $\Omega = 0$ because $\bar{v}(\vec{x}, -\Omega) = \bar{v}(\vec{x}, \Omega)$ [2].

Using the explicit expression of the amplified input field (25) and the fact that $\bar{u}(\vec{x}, \Omega) = \bar{u}(-\vec{x}, \Omega)$, $\bar{v}(\vec{x}, \Omega) = \bar{v}(-\vec{x}, \Omega)$ and Eq.(5), we find for the zero frequency value of the spectrum

$$V_{i_-}(0) = (1 - \eta)\langle i_+ \rangle + \eta^2 \int_{R_1+R_2} d\vec{x} |\alpha_{in}(\vec{x})|^2 \quad (36)$$

As shown in [2] in the case of ideal detection ($\eta = 1$), the noise level of i_- reduces therefore to the noise of the input image over $R_1 + R_2$. As a consequence, under conditions of large gain, fluctuations are well below the shot noise level.

It is important now to connect with the result for quadrature components obtained in the previous section. To this aim, let us first assume that the input field is strictly different from zero at least in some region of the transverse plane. Second, let us assume that the parametric values are such that the pure noise contribution in $V_{i_-}(\Omega)$ (i.e. the last term in Eq.(35)) is negligible and, similarly, that the second term in Eq.(11) can be dropped. Thus expression (35) reduces to

$$\begin{aligned}
V_{i_-}(\Omega) = & (1 - \eta)SN_- \\
& + \eta^2 \int_{R_1+R_2} d\vec{x} |\bar{u}(\vec{x}, \Omega)\alpha_{out}^*(\vec{x}) - \bar{v}^*(\vec{x}, -\Omega)\alpha_{out}(-\vec{x})|^2, \quad (37)
\end{aligned}$$

where

$$SN_- = \eta \int_{R_1+R_2} d\vec{x} |\alpha_{out}(\vec{x})|^2 \quad (38)$$

and we used Eq.(25). By comparing with Eqs.(19) and (20), we see that this expression coincides with $V_{\phi_L}^{(-)}(\Omega)$ if we take:

$$\alpha_L(\vec{x}) = \alpha_{out}(\vec{x}) . \quad (39)$$

This is expected because a LO with the configuration of the output field just picks up the amplitude fluctuations. The relation becomes even more precise in the phase-sensitive case $\alpha_{in}(\vec{x}) = \alpha_{in}(-\vec{x})$. In this case, assuming $\eta = 1$ and that the pixel area is small with respect to x_0^2 and to the square of the scale of variation of $\alpha_{out}(\vec{x})$, one has:

$$\frac{V_{i-}(\Omega)}{SN_-} = \tilde{F}(\vec{x}, \Omega) = |\bar{u}(\vec{x}, \Omega)e^{-i\phi_{out}(\vec{x})} - \bar{v}^*(\vec{x}, -\Omega)e^{i\phi_{out}(\vec{x})}|^2 , \quad (40)$$

where we set $\alpha_{out}(\vec{x}) = \rho_{out}(\vec{x})\exp(i\phi_{out}(\vec{x}))$, and \vec{x} is the central point of any of the two symmetrical pixels. The result coincides with that of Eq.(21) where ϕ_L is replaced by ϕ_{out} . The link between intensity fluctuations and quadrature fluctuations allows now to analyse immediately the case of phase fluctuations, which coincide with the quadrature fluctuations obtained by using a LO which displays a phase shift of $\pi/2$ with respect to LO which provides the amplitude fluctuations. Therefore, in terms of pixels, we are lead consider the observables $Z_{\phi_L+\pi/2}^{(j)}(t)$ (see Eq.(16)) with $\phi_L = \phi_{out}$. This naturally induces us to focus on the observable $Z^{(+)}(t) = Z_{\phi_{out}+\pi/2}^{(1)}(t) + Z_{\phi_{out}+\pi/2}^{(2)}(t)$, see Eq.(17), which measures the degree of anticorrelation between the phase fluctuations in the two symmetrical pixels 1 and 2. The spectrum $V_{\phi_L+\pi/2}^{(+)}(\Omega)$ coincides with $V_{\phi_L}^{(-)}(\Omega)$, which as we have seen is identical to $V_{i-}(\Omega)$ given by Eq.(40). Therefore for large amplification the fluctuations of $Z^{(+)}(t)$ are well below the shot noise level, which implies that the phase fluctuations in the two symmetrical pixels are strongly anticorrelated, exactly as the amplitude fluctuations are strongly correlated.

V. CONCLUSION

In this article we analyzed extensively a system formed by an optical parametric amplifier with some imaging lenses. Amplification of optical images by OPA has been already studied in the literature [15], but only from a classical viewpoint.

Our results hold both for a phase-sensitive configuration (symmetrical input image) and for a phase insensitive one (asymmetrical injection).

We demonstrated that the two output twin images exhibit a complete spatial EPR entanglement. This was shown, first of all, by considering a pair of orthogonal quadrature

components of the output field. In the case of local oscillator symmetrical with respect to the system axis, we found a precise prescription for the phase ϕ_L of the local oscillator (see introduction) in order to observe maximal correlation between symmetrical pixels of the two output images. The optimal value for the phase is that which corresponds to the amplitude fluctuations of the output images in the phase sensitive configuration, when the phase of the symmetrical input images is selected to obtain maximal amplification.

We have shown also that a performance very close to that of the ideal case of the optimal LO phase can be obtained in practice by using a LO with a quadrature wavefront (as one has in gaussian beams) with the curvature used as optimisation parameter.

The connection between quadrature fluctuations and amplitude/phase fluctuations has in turn allowed us to conclude also that, while intensity fluctuations are strongly correlated in the twin images, phase fluctuations are strongly anticorrelated in the same amount. An amusing analogy with amplitude and phase fluctuations in entangled twin images is provided by a fossile broken in two pieces (see Fig.3). We see that the structures in the two pieces have the same “amplitude” = thickness, but opposite “phase” (one is concave and the other convex, one is righthanded and the other lefthanded).

It is important to underline that, while the results for intensity and phase fluctuations hold only in presence of an input image, the result on EPR entanglement of quadrature components hold also in absence of any input image, i.e. in the case of the pure parametric down-conversion as in [14]. This is important for the applications to quantum teleportation of optical images [16], as a generalisation of the Braunstein-Kimble [17, 18] scheme for a single mode field, or to quantum cryptography with images.

We observe finally that our results hold also if the OPA is replaced by an optical parametric oscillator below threshold with plane mirrors (see [2] in this connection). As a matter of fact, also in this case one has an input-output relation of the form (2), and the results are based only on this relation and on the general properties of the functions \bar{u} and \bar{v} .

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VI. FIGURE CAPTIONS

FIG. 1: Schematic diagram of the parametric image amplifier. The two-lens telescopic system allows to obtain two amplified copies of the input image that are strongly quantum correlated to each other, thereby the name twin images. The device is phase-sensitive when the input image is symmetrical, phase-insensitive when it is confined in the upper half of plane P_1 . f is the focal distance of the lenses.

FIG. 2: Plot of the noise reduction factor $F(\vec{x}, 0)$. Subscripts (a) refers to the optimal phase of the LO while (b) and (c) refer respectively to a constant phase and to a phase with a quadratic dependence on the distance from the optical axis. The dashed line is the phase-sensitive gain of the OPA (see Eq.(27) divided by a factor 10). $\Delta_0 l_c = 0.5$ and $|\sigma| l_c = 1.5$.

FIG. 3: Analogy between a broken fossile and quantum entangled images (see text).



